Baumjohann, W., and R. A. Treumann,
Basic space plasma physics, Revised edition,
Imperial College Press, 2012.

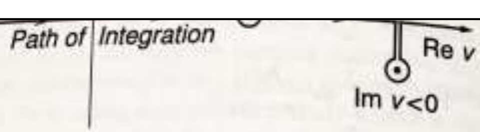

Figure 10.2: Integration contours for three positions of the pole. In this case the integration is again right-handed along the real axis. For inclusion of poles, on the real axis and below it must be deformed accordingly. The pole on the real axis contributes half the residuum, while that below contributes a full residuum. For the positive pole the path needs no to be deformed.

a small correction to p. The integral of the first term in Eq. (10.30) is the unperturbed density, n_0. The integral over the second term vanishes because the plasma is at rest, $\langle v \rangle = 0$. The integral over the third term is $n_0 k_B T_e / 2m_e$. Keeping only these three terms and dropping the small contribution of the pole, the real part, $\epsilon_r(\omega, k)$, of the dispersion relation $\epsilon(\omega, k)$ takes the form

$$\epsilon_r(\omega,k) \equiv 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{3\omega_{pe}^2}{\omega^2}\frac{k^2 v_{the}^2}{\omega^2} = 0. \tag{10.31}$$

This is the dispersion relation of Langmuir waves given in Eq. (9.32) for the one-dimensional case, $\gamma_e = 3$, if we replace one of the ω^2 in the denominator of the third term by ω_{pe}^2. This term turns out to be the thermal correction to the Langmuir waves and the dispersion relation, including the damping term, becomes

$$\boxed{\omega_l = \pm\omega_{pe}\left(1 + \frac{3}{2}k^2\lambda_D^2\right) + i\gamma_l(k).} \tag{10.32}$$

The collisionless damping decrement can now be determined by inserting $p = i\omega$ into the imaginary correction of Eq. (10.30) and calculating the derivative of the Maxwellian. Under the assumption that the damping is weak, $\gamma_l \ll \omega_l$, it is reasonable to expect that the imaginary part of $\epsilon(k,\omega)$ is also small.

10.2.2 Damping Rate

Under this condition it is possible to develop a simple prescription to determine the damping rate from the dispersion relation. Let us split

$$\epsilon(k,\omega,\gamma) = \epsilon_r(k,\omega,\gamma) + i\,\epsilon_i(k,\omega,\gamma) \tag{10.33}$$

into real and imaginary parts, $\epsilon_r(k,\omega,\gamma)$, $\epsilon_i(k,\omega,\gamma)$. Since $\gamma_l \ll \omega_l$, one can expand with respect to the real frequency at vanishing damping rate and obtain

$$\epsilon(k,\omega,\gamma) = \epsilon_r(k,\omega,0) + i\gamma \left.\frac{\partial \epsilon_r(k,\omega,\gamma)}{\partial \omega}\right|_{\gamma=0} + i\epsilon_i(k,\omega,0) = 0. \tag{10.34}$$

Setting the real and imaginary parts of this expression separately equal to zero gives

$$\epsilon_r(k,\omega,0) = 0, \tag{10.35}$$

$$\gamma(k,\omega) = -\frac{\epsilon_i(k,\omega,0)}{\partial \epsilon_r(k,\omega,\gamma)/\partial\omega|_{\gamma=0}}. \tag{10.36}$$

The first of these equations is the usual dispersion relation depending only on real quantities. But the second equation is a very useful expression for calculating the damping rate of any weakly damped wave. In the following we will make extensive use of this expression. Applying it to the Langmuir wave dispersion relation Eq. (10.30) yields

$$\boxed{\gamma_l(k) = -\left(\frac{\pi}{8}\right)^{1/2} \frac{\omega_{pe}}{k^3\lambda_D^3} \exp\left(-\frac{1}{2k^2\lambda_D^2} - \frac{3}{2}\right)} \tag{10.37}$$

as an expression for the Landau damping of high-frequency Langmuir waves in a collisionless unmagnetised plasma. This damping is not caused by particle collisions but is entirely due to particle decorrelation effects.

10.2.3 Physics of Landau Damping

The appearance of collisionless dissipation in the dispersion of electron plasma waves is somewhat disturbing, since dissipation implies a preferred direction in time, while the Vlasov equation is reversible in time, as can be seen by substituting $t \to -t$ and $v \to -v$ into Eq. (10.1). However, Landau damping only affects a very small part of the distribution function. Only particles with velocities close to the phase velocity of the wave, $v_{ph} = \omega/k$ become resonant and are redistributed in phase space during their interaction with the wave. Hence, the directivity does not affect most of the distribution function and thus has no effect on the time symmetry of the Vlasov equation.

In order to understand the physics of Landau damping, let us consider a plasma wave propagating across a plasma in thermal equilibrium with a Maxwellian equilibrium distribution function, $f_0(v)$. The situation is depicted in the left-hand part of Fig. 10.3. Particles at position $v = v_{ph} = \omega/k$ are in resonance with the wave, since they are moving at the same speed as the wave in the plasma. Clearly, these particles interact strongest with the wave electric field. Depending on the direction of this field, the particles will either be accelerated or decelerated. On the other hand, any particles moving slightly faster or slower than the wave will experience a different kind of interaction.

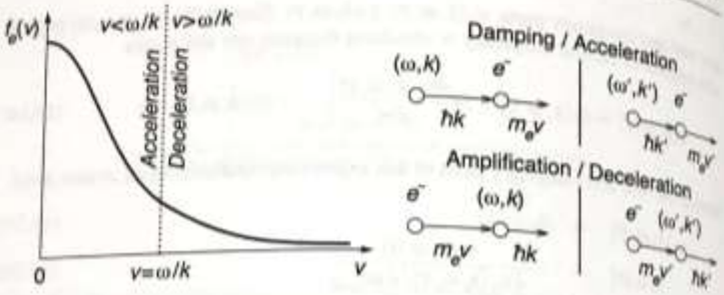

Figure 10.3: The mechanism of Landau damping. *Left*: The wave resonance is at location $v = \omega/k$. All electrons left of it are slow, those at the right are faster than the wave. Accordingly, in interaction, the faster electrons will give up energy to the wave while the slower will absorb energy from the wave. *Right*: This is shown in analogy of photon-electron collisions where a wave of frequency ω and wave number k has energy $\hbar\omega$ and momentum $\hbar k$.

We can investigate this kind of interaction by exploiting the simple analogy of a collision between two particles (right-hand side of Fig. 10.3), taking the wave as an uncharged particle of energy $\hbar\omega$ and momentum $\hbar k$. In a collision between the two particles (electron and wave), the one with the higher momentum will always speed up the lower momentum particle, thereby losing its own energy.

This kind of interaction is of elastic nature and thus dissipationless. It exactly resembles the process of Landau damping as an elastic interaction between particles and waves with no preferred direction. Fast electrons will speed up the waves, while slow electrons are pushed by the wave and gain energy. But why then an effective damping of the wave? The reason is the asymmetry of the Maxwellian distribution function with respect to the plasma wave phase velocity. There are more low than high velocity particles, and, hence, the wave loses more momentum and energy in the interaction with low momentum particles than it gains back from interaction with higher momentum particles. Clearly, during this process the distribution function must necessarily become slightly distorted, as shown in Fig. 10.4. The retarded and accelerated particles right and left of the resonance are attracted by the resonance and accumulate there.

We can make this argument a little more quantitative by estimating the change in energy which the electron distribution experiences during the interaction with a Langmuir wave. This change is given by the integral

$$\delta W_e = m_e \int_{-\infty}^{\infty} dv\, v \langle \Delta v \rangle f(v), \tag{10.38}$$

where $\langle \Delta v \rangle$ is the electron velocity change averaged over one Langmuir wavelength. It is convenient to transform to a system moving with phase velocity ω/k, choosing

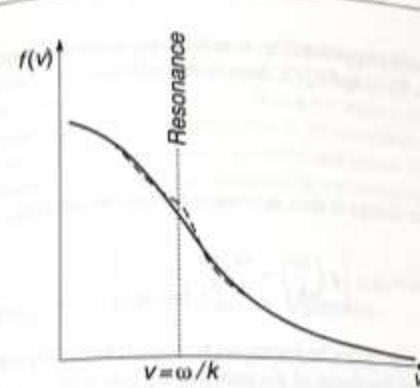

Figure 10.4: Attraction of particles by a resonance. The energy exchange retards the faster and accelerates the slower electrons thus 'attracting' them from both sides and generating a plateau on the distribution function around the resonance. This also causes some steeping of the distribution function on both sides outside the resonance. Resonant waves thus cause a local distortion on a thermal distribution function in velocity space, which in principle implies a source of free thermal energy which may become available for other processes.

$v' = v - \omega/k$. The change in velocity, $\langle \Delta v \rangle$, is independent of such a transformation, and the above integral becomes

$$\delta W_e = m_e \int_{-\infty}^{\infty} dv' \left(v' + \frac{\omega}{k}\right) \langle \Delta v \rangle f\left(v' + \frac{\omega}{k}\right). \tag{10.39}$$

We now expand the distribution function around $v' = 0$ to obtain

$$\delta W_e = m_e \int_{-\infty}^{\infty} dv' \left(v' + \frac{\omega}{k}\right) \langle \Delta v \rangle \left[f\left(\frac{\omega}{k}\right) + v' \left.\frac{\partial f}{\partial v}\right|_{v'=\omega/k} \right]. \tag{10.40}$$

It can be shown by using the electron equation of motion in an oscillating electric wave field of amplitude δE_0,

$$m_e \frac{dv_e}{dt} = -e\delta E_0 \sin\left[kx(t) - \omega t\right], \tag{10.41}$$

that the average velocity variation, $\langle \Delta v \rangle$, is an odd function of v' (calculate $v_e(t), x(t)$ from the above equation of motion and substitute $x(t) = x_0 + v't$ back to determine Δv). Hence, of the four product terms in the above integral only the two terms containing the product $v'\langle \Delta v \rangle$ survive the integration from $-\infty$ to $+\infty$. After integrating Δv over one oscillation period, one finds

$$\langle \Delta v(t) \rangle = \frac{A}{k^2 v'^3} \left[\cos(kv't) - 1 + \frac{1}{2}(kv't)\sin(kv't) \right]. \tag{10.42}$$

Between the constant of proportionality, A, and the wave energy density averaged over one wave oscillation, $W_w = \varepsilon_0\,\delta E_0^2/2$, there is the relation

$$A = W_w \frac{\omega_{pe}^2}{n_0 m_e}. \tag{10.43}$$

The change in particle energy is thus determined from the integral

$$\Delta W_e = m_e \int_{-\infty}^{\infty} dv' v' \langle \Delta v \rangle \left[f\left(\frac{\omega}{k}\right) + \frac{\omega}{k} \frac{\partial f}{\partial v'}\bigg|_{v'=\omega/k} \right]. \tag{10.44}$$

The first term in the brackets can be neglected because it adds only a small contribution which is independent of the shape of the distribution function:

$$\Delta W_e \approx \frac{m_e \omega}{k} \frac{\partial f}{\partial v'}\bigg|_{v'=\omega/k} \int_{-\infty}^{\infty} dv' v' \langle \Delta v \rangle. \tag{10.45}$$

After integration and replacing $\omega \approx \omega_{pe}$, we get

$$\Delta W_e(t) = -A\pi t \frac{m_e \omega_{pe}}{k^2} \frac{\partial f}{\partial v}\bigg|_{v=\omega_{pe}/k}. \tag{10.46}$$

Hence, in the average over one wave oscillation period the electrons gain energy, if the derivative of the equilibrium distribution function in the vicinity of the resonance is negative. This energy is provided by the wave and leads to acceleration of the small number of resonant particles with velocities just below the wave phase speed. In equilibrium the energy transferred to the electrons per unit time equals the loss of wave energy:

$$\frac{\Delta W_e(t)}{\Delta t} = -\frac{dW_w(t)}{dt} = 2\gamma_l W_w(0)\exp(-2\gamma_l t). \tag{10.47}$$

The second part of this equation results from the definition of the average wave energy $W_w = \varepsilon_0 \delta E(t) \cdot \delta E^*(t)/2$, where, after multiplication of the wave electric field with its conjugate complex part, only twice the real part of the exponent survives. Inserting Eq. (10.46) for the gain in electron energy and Eq. (10.43) for the amplitude factor, A, one finds that the wave energy, W_w, appears on both sides of the equation and obtains another expression for the Landau damping:

$$\gamma_l = \omega_{pe} \frac{\pi \omega_{pe}^2}{2 n_0 k^2} \frac{\partial f}{\partial v}\bigg|_{v=\omega_{pe}/k}. \tag{10.48}$$

Inserting the Maxwellian distribution and remembering that it is normalised to the unperturbed density, n_0, one just recovers Eq. (10.37).

Equation (10.48) confirms that the derivative of the equilibrium distribution function in the vicinity of the resonance decides about the sign of the real part of p. Typically the derivative is negative, we have $\gamma_l < 0$ and the wave is damped. However, if the distribution function has a positive derivative at the resonance, we have $\gamma_l > 0$ and the wave extracts energy from the resonant particles and grows. Such *inverse Landau damping* implies instability and will be discussed in our companion book, *Advanced Space Plasma Physics*.

10.3 Unmagnetised Plasma Waves

Landau damping does not only affect the Langmuir waves, but also the other wave modes that propagate in a warm unmagnetised plasma. In addition, a new wave mode appears in the kinetic treatment.

10.3.1 Ion-Acoustic Waves

So far we have suppressed the contribution of ion inertia. From the derivation of the dispersion relation, $\epsilon(k,p) = 0$, it has become clear that the contributions of different species (electrons, ions, etc.) can be accounted for by adding a singular integral over the distribution function of the corresponding species, of the same kind as in Eq. (10.27), to $\epsilon(k,p)$. Hence, including the ion contribution requires solving the following dispersion relation:

$$\epsilon(k,p) = 1 + \frac{\omega_{pe}^2}{n_{0e}k^2} \int_{-\infty}^{\infty} \frac{dv\, f_{0e}(v)}{(v - ip/k)^2} + \frac{\omega_{pi}^2}{n_{0i}k^2} \int_{-\infty}^{\infty} \frac{dv\, f_{0i}(v)}{(v - ip/k)^2} = 0. \tag{10.49}$$

At the high frequencies corresponding to electron plasma oscillations we can use the same expansion for the two integrals as before and find for the real part,

$$1 - \frac{\omega_{pe}^2 + \omega_{pi}^2}{\omega^2} - \frac{3k^2}{\omega^4}\left(\omega_{pe}^2 v_{the}^2 + \omega_{pi}^2 v_{thi}^2\right) = 0, \tag{10.50}$$

which leads to a slightly modified dispersion relation of Langmuir waves, corrected for the effect of ions:

$$\boxed{\omega_l^2(k) = \omega_{pe}^2 \left(1 + \frac{m_e}{m_i}\right)\left[1 + \frac{3k^2\lambda_D^2}{1 + m_e/m_i}\left(1 + \frac{m_e^2}{m_i^2}\frac{T_i}{T_e}\right)\right].} \tag{10.51}$$

The difference between the simple Langmuir dispersion relation and this corrected version is small. The plasma frequency is corrected by a term of the order of the electron-to-ion mass ratio. The correction of the Debye length turns out even smaller and becomes important only for extremely high ion temperatures.